Existence of dominant players and their role in the formation of a cabinet coalition

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Abstract

A party is dominant if there is a majority coalition to which that party belongs such that it affords this party more possibilities to form an alternative winning coalition than any of the other members of the coalition (see Peleg [1980, 1981]). In this article I present empirical evidence showing that an allocation of seats in a parliament is biased toward the high frequency occurrence of a dominant party and the low frequency occurrence of a dictator (one party that holds a majority). If a dominant party forms a cabinet coalition, and if that cabinet coalition has a majority in parliament, then with a significantly high frequency the dominant party tends to form a coalition which it dominates. Evidence indicates that the occurrence of a dominant party increases as the number of parties in parliament increases.

1 Introduction

There are different criteria for measuring the quality of a political system. In this paper I argue that one of desired properties of a multi-party system is the existence of one party that is “stronger” than any of the others. If such a party exists, it may improve the

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governability of the political system. Applying the game-theory concept of a dominant player (see Peleg [1980, 1981]) to define a “strong” party, a formal definition is given in Section 2, but for now the property is described informally.

There is a number of parties in a parliament. Suppose that party \( i \), together with a coalition of parties \( S \), forms a majority in the parliament. Suppose also that \( i \) has at least one another alternative to form a majority, for instance, to join coalition \( S' \) (no member of \( S \) belongs to \( S' \), and together \( S \) and \( S' \) have no majority). However, if \( S \) can form a majority without party \( i \), for example, together with coalition \( T \), then also when \( i \) joins \( T \), it obtains the majority. When the above holds, \( i \) is a dominant party, and it dominates the coalition \( \{i\} \cup S \). Intuitively, party \( i \) has more freedom to form a majority coalition without its partners in \( S \) while members of \( S \) have fewer possibilities to form a majority. In this sense \( i \) is stronger than the other parties. It has been proven by Peleg [1981] that if only a simple majority is required, there exists at most one party with such a property in a parliament (see Proposition 2.7).

A special case of a dominant party is a party which has a majority of seats in parliament and therefore, does not need any other party to form a majority coalition. In this paper such a party is called a "dictator". Obviously, the existence of a dictator improves the governability of the political system. On the other hand, if a dictator party exists, the ability of other parties to influence the outcome of political decision-making vanishes. It may be suggested that an optimal allocation of seats in a parliament is one in which one party is stronger than all the others (in terms of this paper, it is called dominant), but it is not strong enough to be a dictator.

Using empirical data about the allocation of seats in parliaments from 15 countries (see Appendix 1 for a description of the data), I found that out of 336 observations (compositions of parliaments), 257 had a dominant party while only 32 had a dictator. My claim is that this frequency of dominated parties is significantly high and that the frequency of dictator parties is significantly low. The following statistical test is used (see Section 3 for a detailed description). For every observation I assume that parties represented in parliament are the only political actors; that is, parties that do not obtain
any seats in parliament can be ignored. Let seats be allocated to those parties randomly, where all compositions of parliaments are equally likely. The probability that there is a dominant party or that there is a dictator is computed. After repeating this procedure 336 times, for each real-data observation, I found that the probability of randomly obtaining 257 or more parliaments containing a dominant party, or obtaining 32 or less parliaments containing a dictator, is negligibly low. Thus, a distribution of parliament compositions, at least in the data sample used, is biased toward the high frequency occurrence of a dominant party and the low frequency occurrence of a dictator.

It may be interesting to perform the analysis described above by each one of countries. Unfortunately, a number of elections in most countries is not too high to provide a sufficiently large sample for statistical analysis. Nevertheless, if only countries with number of observations above 20 are considered, significant results similar to the general (cross-country) one are obtained for the Netherlands, for Denmark and for Switzerland.

The question of government coalition formation is widely discussed in the literature (see Riker [1962], de Swaan [1973], Axelrod [1970], Lijphart [2012, Chapter 6] and Chua and Felsenthal [2008]). I suggest that the strength of parties, in the game-theoretic sense, is one of factors in the formation of coalitions.

Suppose that a dominant party exists. First, in my data sample, in 305 out of 367 cases the dominant party is chosen to form a cabinet. But this finding is not significant since the dominant party has to be the largest one (Peleg [1981]). Therefore, it is not clear if it was chosen to form a cabinet because of its being a dominant party or because of its size. Next we take into account only parliamentary compositions and cabinet coalitions where a dominant party exists and it is not a dictator and it forms the coalition and the cabinet coalition has the majority. By definition, a party is dominant if there is a coalition that it dominates, and the party forms the majority by joining that coalition. But will the dominant party actually use its theoretical advantage to form a coalition that it dominates? I found that out of the 129 cabinet coalitions in the dataset 90 were dominated. If coalitions are chosen randomly with a uniform distribution, out of all majority coalitions including dominant parties, the probability of obtaining 90 or more
dominated coalitions is close to zero. This supports the hypothesis that if a dominant party forms a majority coalition, with a significantly high frequency it tends to chose a coalition that it dominates. See Section 4 for a detailed description.

How is frequency of a dominant party related to the number of parties in a parliament? The intuitive answer is not unequivocal. On the one hand, if the number of parties decreases one can expect that the relative size of the largest party will increase and so will the probability that it is dominant. On the other hand, when the number of parties increases, the number of possible coalitions also increases, so it may be expected that the probability that the largest party is dominant increases. Statistical analysis of the data (Section 5) provides evidence that if the number of parties is relatively low, the frequency of a dominant party increases with the number of parties. While I conjecture that this property does not hold for a sufficiently high number of parties, there is not enough data to give it support.

Note that the definition of a dominant party is based on the whole composition of the parliament. That is the reason I use the number of parties as it was in a parliament. Other measures of number of parties, like the ”effective number” (Laakso and Taagepera [1979]) can not be applied to this model.

The notion of ”dominant party” used in the political science literature has a different meaning than the one used here. I now discuss the relation between the two.

Duverger [1959] in his classic text wrote: ”What is a dominant party? First of all a party larger than any of the others and which heads the list and clearly out-distances its rivals over a certain period of time”. Note that the ”game-theory” definition of dominant party used in this work refers to a distribution of seats to parties at a single time, and not to any prolonged tendency. Duverger continues with a ”softer” definition: ”Every party that is larger than all the others over a certain period of time is not necessarily dominant .... A party is dominant when it is identified with an epoch .... A dominant party is that which public opinion believes to be dominant”. Similarly, Sartori [1976] argues that ”Whenever we find, in a polity, a party that outdistances all the others, this party is dominant in that it is significantly stronger than all the others”. Sartori also
defines a predominant party as one which obtains a majority of seats in a number of sequential elections. This is a special case of what I have called a "dictator". Note that the definition of a dictator party does not require a majority in a number of parliaments. Pempel [1990] suggests a detailed list of conditions for being a dominant party in the Duverger-Sartori sense:

(C1) **Dominant in number**: it must win a larger number of seats than its opponents.

(C2) **Dominant bargaining position**: it must be in a strategic position that makes it highly unlikely for any government to be formed without its inclusion.

(C3) **Dominant chronologically**: it must be at the core of a nation’s government over a substantial period of time.

(C4) **Dominant governmentally**: the dominant party carries out what many would call a historical project that gives a particular shape to the national political agenda.

Actually, an intuition of the “game-theory” definition of a dominant party used in this paper is compatible with (C2) when (C1) is obtained as a result following from the dominant party definition (Proposition 2.7). But Pempel’s requirements for a dominant party are stronger because of (C3) and (C4), which are beyond the scope of the current work. Therefore Pempel’s notion can be interpreted as a special case of the concept of this paper.

It should be emphasized that in this work parties’ ideological stances are not taken into account. A coalition is defined as winning if it has a majority of seats. Obviously, in reality some majority coalitions cannot occur because of ideological differences between its members. The definition of dominant party I use here holds for the general class of simple monotonic games, but in examining its empirical application I chose ”objective”, technical data only and ignore ideological issues which can be interpreted subjectively.

Moreover, I assume that there are no dependencies between different election outcomes. A more complicated model, where election outcome depends on the previous election, is left for the further research.
2 Model

For \( n \in \mathbb{N} \), let \([n] = \{1, \ldots, n\}\) be the set of parties.

**Definition 2.1.** A **simple game** is an ordered pair \( G = ([n], W) \), where \( n \in \mathbb{N} \), \([n]\) is the set of players and \( W \) is a set of coalitions (subsets of \([n]\)) whose members are in the **winning coalition**. \( G \) is **monotonic** if

\[
S \in W \text{ and } S \subset T \Rightarrow T \in W
\]

**Definition 2.2.** A simple game \( G \) is **proper** if

\[
S \in W \Rightarrow [n] \setminus S \not\in W
\]

Let \( G = ([n], W) \) be a monotonic simple game.

**Definition 2.3.** Coalition \( S \subseteq N \) is **at least as desirable as** \( T \subseteq N \) (denoted by \( S \succeq T \)), if for every \( B \subseteq [n], B \cap (S \cup T) = \emptyset, T \cup B \in W \Rightarrow S \cup B \in W \).

**Definition 2.4.** \( i \in S \) **dominates** \( S \) if \( \{i\} \succeq S \setminus \{i\}, \) but \( S \setminus \{i\} \not\succeq \{i\} \).

**Definition 2.5.** A player \( i \in N \) is **dominant**, if there is a coalition \( S \subseteq N, i \in S, S \in W, \) such that \( i \) dominates \( S \).

The special interest in this paper is the class of monotonic simple games named "weighted majority games".

**Definition 2.6.** Let \([n]\) be the set of parties, \( n \in \mathbb{N} \). Let \( w \in \mathbb{R}^n, w_i \geq 0 \) for each \( i \in [n] \) be the vector of weights given to the parties. Let \( 0 < q < \sum_{i \in [n]} w_i \) be the majority quota. A **weighted majority game** \( G([n], w, q) = ([n], W(w, q)) \) is a simple game defined by:

For every \( S \subset [n] \)

\[
\begin{cases} 
S \in W, & \sum_{i \in S} w_i > q \\
S \not\in W, & \text{otherwise}.
\end{cases}
\]

Hereafter I assume a **simple majority**, namely, \( q = \frac{\sum_{i \in [n]} w_i}{2} \). Note that for this majority quota the weighted majority game \( G([n], w, q) \) is proper. Denote

\[
q^*(n, w) = \frac{\sum_{i \in [n]} w_i}{2}.
\]

The following result is due to Peleg [1981]:
Proposition 2.7. Let \( n \in \mathbb{N}, w \in \mathbb{R}^n, w_i \geq 0 \) for each \( i \in [n] \). In a properly weighted majority game \( G([n], w, q^*(n, w)) \) there is at most one dominant player. If it exists, it is player \( i \) which has the largest weight \( w_i \).

Since the current paper is devoted to an application of this theory to parliaments, in this paper the terms "player" and "party" are equivalent.

Definition 2.8. Let \( G(N, w, q) \) be a weighted majority game. Player (party) \( i \) is a dictator if \( w_i > q \).

Note that a dictator is a special case of a dominant party.

3 Frequency of dominant party and of dictator

I now show that the frequency of a dominant player in a real-world parliament is significantly high and the frequency of a dictator is significantly low.

Consider outcomes of real elections, namely compositions of parliaments from 15 countries. For a description of the data, see Appendix 1.

Consider \( M \) different election outcomes (parliamentary compositions). Let \( \Lambda \in \mathbb{N}^M \).

For every \( m \in [M] \) let \( \Lambda(m) \) be the number of parties represented in a parliament in the \( m \)th outcome.

Let \( \mathcal{P} = (P_1, \ldots, P_M) \) be the vector of election outcomes, each \( P_m, 1 \leq m \leq M \) is a vector of weights, \( P_m = (w_1^m, \ldots, w_{\Lambda(m)}^m) \), \( w_i^m \in \mathbb{N} \) for every \( 1 \leq i \leq \Lambda(m) \). Thus, for every \( 1 \leq m \leq M \), \( P_m \) corresponds to the weighted majority game \( G([\Lambda(m)], P_m, q^*(\Lambda(m), P_m)) \).

Denote by \( \Delta(\mathcal{P}) \) the number of weighted majority games corresponding to members of \( \mathcal{P} \) with a dominant player. Namely,

\[
\Delta(\mathcal{P}) = |\{m \in [M] \mid \text{in the game } G([\Lambda(m)], P_m, q^*(\Lambda(m), P_m)) \text{ there is a dominant player}\}|
\]

I use the following statistical test to show that \( \Delta(\mathcal{P}) \) is significantly higher than if weights of the weighted majority game were randomly allocated with uniform distribution.

Suppose that in each election weights are randomly allocated to parties according to some unknown distribution. Namely, for each \( m \in [M], P_m \) is a realization of this
random allocation. Let \( \pi^{'\text{DOM}}(m) \) be the probability that in the \( m \)th election outcome, \( m \in [M], \mathcal{P}_m \), there is a dominant party (there is a dominant party in the weighted majority game \( G([\Lambda(m)], \mathcal{P}_m, q'(\Lambda(m), \mathcal{P}_m)) \)). In other words, let existence of a dominant party in \( G([\Lambda(m)], \mathcal{P}_m, q'(\Lambda(m), \mathcal{P}_m)) \) be interpreted as the realization of success in the Bernoulli trial with parameter \( \pi^{'\text{DOM}}(m) \). After \( M \) trials with parameters \( \pi^{'\text{DOM}}(1), \ldots, \pi^{'\text{DOM}}(M) \), the observed number of successes is \( \Delta(\mathcal{P}) \).

For \( n \in \mathbb{N} \), let \( \Omega(n) = \{(w_1, \ldots, w_n) | \sum_{i=1}^n w_i = 1 \} \) be the \((n - 1)\)-dimensional simplex. Let vector \( w \) be picked from \( \Omega(n) \) with uniform distribution. I define then \( \pi^{\text{DOM}}(n) \) as the probability that in game \( G([n], w, q'(n, w)) \) there is a dominant player. Formally, denote

\[
\Omega^{\text{DOM}}(n) = \{ w \in \Omega(n) | \text{in the game } G([n], w, q'(n, w)) \text{ there is a dominant player} \},
\]

Then

\[
\pi^{\text{DOM}}(n) = \text{Prob}(w \in \Omega(n)^{\text{DOM}} | w \in \Omega(n)),
\]

when \( w \in \Omega(n) \) is drawn with uniform distribution.

The following hypotheses about \( \mathcal{P} \) are to be tested:

\( (H0^{\text{DOM}}) \) For every \( m \in [M], \pi^{'\text{DOM}}(m) = \pi^{\text{DOM}}(\Lambda(m)) \).

\( (H1^{\text{DOM}}) \) There exists \( m \in [M] \) such that \( \pi^{'\text{DOM}}(m) > \pi^{\text{DOM}}(\Lambda(m)) \).

Let

\[
\pi^{\text{DOM}}(\Lambda) = (\pi^{\text{DOM}}(\Lambda(1)), \ldots, \pi^{\text{DOM}}(\Lambda(M)))
\]

be the vector that defines a sequence of \( M \) independent Bernoulli trials, when \( \pi^{\text{DOM}}(\Lambda(m)) \) is the parameter of the \( m \)th trial, \( m \in [M] \). Denote by \( X(\pi^{\text{DOM}}(\Lambda)) \) a random variable of the number of successes in \( M \) independent Bernoulli trials with parameters given by \( \pi^{\text{DOM}}(\Lambda) \).

\( (H0^{\text{DOM}}) \) will be rejected and \( (H1^{\text{DOM}}) \) will be accepted if \( \text{Prob}(X(\pi^{\text{DOM}}(\Lambda)) \geq \Delta(\mathcal{P})) < 0.05 \). I proceed next to estimate \( \text{Prob}(X(\pi^{\text{DOM}}(\Lambda)) \geq \Delta(\mathcal{P})) \). By the Central Limit Theorem,

\[
\frac{X(\pi^{\text{DOM}}(\Lambda)) - \mu^{\text{DOM}}(\pi^{\text{DOM}}(\Lambda))}{\sigma^{\text{DOM}}(\pi^{\text{DOM}}(\Lambda))} \xrightarrow{D} \mathcal{N}(1, 0), \text{ as } M \to \infty, \tag{3.1}
\]
where
\[ \mu^{\text{DOM}}(\pi^{\text{DOM}}(\Lambda)) = \sum_{m=1}^{M} \pi^{\text{DOM}}(\Lambda(m)), \] (3.2)
\[ (\sigma^{\text{DOM}}(\pi^{\text{DOM}}(\Lambda)))^2 = \sum_{m=1}^{M} \pi^{\text{DOM}}(\Lambda(m))(1 - \pi^{\text{DOM}}(\Lambda(l))) \] (3.3)
and \( N(1, 0) \) is the standard normal distribution.

(3.1) is used to estimate \( \text{Prob}(X(\pi^{\text{DOM}}(\Lambda)) \geq \Delta(\mathcal{P})) \).

Let \( \mathcal{P} \) be the set of real-data election outcomes (for the data description see Appendix 1. For its summary see Appendix 2, Table 5). In this case, \( M = 336 \) and \( \Delta(\mathcal{P}) = 257 \).

Estimations for \( \pi^{\text{DOM}}(n), \pi^{\text{DIC}}(n), n = 1, \ldots, 17 \) are summarized in Appendix 2, Table 5.

Using (3.2) and (3.3):
\[ \mu^{\text{DOM}}(\pi^{\text{DOM}}(\Lambda)) = 237.5, \quad \text{(3.4)} \]
\[ \sigma^{\text{DOM}}(\pi^{\text{DOM}}(\Lambda)) = 8, \quad \text{(3.5)} \]

Recall, \( X(\pi^{\text{DOM}}(\Lambda)) \) is the number of successes after \( M \) independent Bernoulli trials with parameters \( \pi^{\text{DOM}}(\Lambda) \). After substitution of (3.4) and (3.5) into (3.1):
\[ \text{Prob}(X(\pi^{\text{DOM}}(\Lambda)) \geq \Delta(\mathcal{P})) = \text{Prob}(X(\pi^{\text{DOM}}(\Lambda)) \geq 257) \approx 0.01 \]

According to these findings hypothesis \((H_0^{\text{DOM}})\) is rejected and \((H_1^{\text{DOM}})\) is accepted.

Similarly, denote by \( \delta(\mathcal{P}) \) the number of games with a dictator among the weighted majority games corresponding to \( \mathcal{P} \). Formally,
\[ \delta(\mathcal{P}) = |\{m \in [M] \text{ in the game } G([\Lambda(m)], \mathcal{P}_m, q^*(\Lambda(m), \mathcal{P}_m)) \text{ there is a dictator}\}| \]

Let \( \pi'^{\text{DIC}}(m) \) be the probability that in the \( m \)th election outcome, \( m \in [M], \mathcal{P}_m \), there is a dictator. The existence of a dictator in \( G([\Lambda(m)], \mathcal{P}_m, q^*(\Lambda(m), \mathcal{P}_m)) \) is the success in the Bernoulli trial with parameter \( \pi'^{\text{DIC}}(m) \). After \( M \) trials with parameters \( \pi'^{\text{DIC}}(1), \ldots, \pi'^{\text{DIC}}(M) \), the observed number of successes is \( \delta(\mathcal{P}) \).

Denote
\[ \Omega^{\text{DIC}}(n) = \{w \in \Omega | \text{ in the game } G([n], w, q^*(n, w)) \text{ there is a dictator}\}, \]
Then
\[
\pi^{DIC}(n) = \text{Prob}(w \in \Omega(n)^{DIC}|w \in \Omega(n)),
\]
when \( w \in \Omega(n) \) is drawn with uniform distribution.

The following hypotheses about \( \mathcal{P} \) are to be tested:

\((H_0^{DIC})\) For every \( m \in [M] \), \( \pi'(DIC)(m) = \pi^{DIC}(\Lambda(m)) \).

\((H_1^{DIC})\) There exists \( m \in [M] \) such that \( \pi'(DIC)(m) < \pi^{DIC}(\Lambda(m)) \).

Denote by \( Y(\pi^{DIC}(\Lambda)) \) the random number of successes in \( M \) independent Bernoulli trials with parameters given by \( \pi^{DIC}(\Lambda) \).

\((H_0^{DIC})\) will be rejected and \((H_1^{DIC})\) will be accepted if \( \text{Prob}(Y(\pi^{DIC}(\Lambda)) \leq \Delta(\mathcal{P})) < 0.05 \).

I estimate \( \text{Prob}(Y(\pi^{DIC}(\Lambda)) \leq \delta(\mathcal{P})) \). By the Central Limit Theorem,
\[
\frac{Y(\pi^{DIC}(\Lambda)) - \mu^{DIC}(\pi^{DIC}(\Lambda))}{\sigma^{DIC}(\pi^{DIC}(\Lambda))} \xrightarrow{D} N(1, 0), \text{ as } M \to \infty, \tag{3.6}
\]
where
\[
\mu^{DIC}(\pi^{DIC}(\Lambda)) = \sum_{l=1}^{M} \pi^{DIC}(\Lambda(l)), \tag{3.7}
\]
\[
(\sigma^{DIC}(\pi^{DIC}(\Lambda)))^2 = \sum_{l=1}^{M} \pi^{DIC}(\Lambda(l))(1 - \pi^{DIC}(\Lambda(l))). \tag{3.8}
\]
As above, \( N(1, 0) \) is the standard normal distribution.

Let \( \mathcal{P} \) be the set of real-data election outcomes (for the data description see Appendix 1. For its summary see Appendix 2, Table 5). In this case, \( M = 336 \) and \( \delta(\mathcal{P}) = 32 \).

Estimations for \( \pi^{DIC}(n), n = 1, \ldots, 17 \) are summarized in Appendix 2, Table 5.

Using (3.7) and (3.8):
\[
\mu^{DIC}(\pi^{DIC}(\Lambda)) = 71.3, \tag{3.9}
\]
\[
\sigma^{DIC}(\pi^{DIC}(\Lambda)) = 6.5. \tag{3.10}
\]
Recall, \( Y(\pi^{DIC}(\Lambda)) \) is the number of successes after \( M \) independent Bernoulli trials \( \pi^{DIC}(\Lambda) \).
After substitution of (3.9) and (3.10) into (3.6):

$$\text{Prob}(Y(\pi^{DIC}(\Lambda)) \leq \delta(P)) = \text{Prob}(Y(\pi^{DIC}(\Lambda)) \leq 32) \approx 0$$

According to these findings hypothesis $H_{0}^{\text{DIC}}$ is rejected and $H_{1}^{\text{DIC}}$ is accepted.

4 Dominant player and cabinet coalition formation

In this section I extend the empirical findings based on Peleg [1980, 1981]. I use a statistical test akin to the one used by de Swaan [1973] and Chua and Felsenthal [2008]. Below is its description.

Consider an allocation of seats in a parliament, such that a dominant, non-dictator party exists. Suppose that this dominant party is appointed by the Head of State to form the cabinet coalition. Suppose that a majority coalition is built. In this section I examine whether the dominant party will "use the advantage" it has and form a coalition that it dominates.

Let us consider $M'$ coalitions. Let $\Lambda' \in \mathbb{N}^{M'}$, $\Lambda'(i)$ be the number of parties represented in parliament for each coalition $1 \leq i \leq M'$.

Let $\mathcal{V} = \{\mathcal{V}_1, \ldots, \mathcal{V}_{M'}\}$, $\mathcal{V}_i \in \mathbb{N}^{\Lambda'(i)}$ be the distribution of seats in parliament for coalition $i \in [M']$.

Let $\mathcal{C} = \{C_1, \ldots, C_{M'}\}$, $C_i \subset \{1, \ldots, \Lambda'(i)\}$ be the coalition, $i \in [M']$. Requirements from $\mathcal{C}$, $\mathcal{V}$ are that:

1. There is a dominant player in the game $G([\Lambda'(i)], \mathcal{V}(i), q^*(\Lambda'(i), \mathcal{V}(i)))$, and this player belongs to $C(i)$ (and it forms the coalition).

2. $C(i)$ is the winning coalition in game $G([\Lambda'(i)], \mathcal{V}(i), q^*(\Lambda'(i), \mathcal{V}(i)))$.

3. There is no dictator in game $G([\Lambda'(i)], \mathcal{V}(i), q^*(\Lambda'(i), \mathcal{V}(i)))$.

Let $\mathcal{F} \in \mathbb{N}^{M'}$. For each $1 \leq i \leq M'$, $\mathcal{F}(i)$ is a dominant player in game $G([\Lambda'(i)], \mathcal{V}(i), q^*(\Lambda'(i), \mathcal{V}(i)))$.

Denote by $D(\mathcal{C})$ the number of coalitions $\mathcal{C}(1), \ldots, \mathcal{C}(M')$, which are dominated by $\mathcal{F}(1), \ldots, \mathcal{F}(M')$, respectively.
Let \( i \in [M'] \). Denote:

\[
W(V(i)) = \{ S \subseteq \{1, \ldots, \Lambda(i)\} | S \text{ as a winning coalition in game } G([\Lambda'(i)], V(i), q^{*}(\Lambda'(i), V(i))) \},
\]

\[
W'(V(i)) = \{ S \in W(V(i)) | J(i) \in W(V(i)) \}
\]

and

\[
W''(V(i)) = \{ S \in W'(V(i)) | J(i) \text{ dominates } S \}
\]

Let \( \pi^C \) be a vector defined by:

\[
\pi^C(i) = \frac{|W''(V(i))|}{|W'(V(i))|}, i \in [M'].
\]

Let \( i \in [M'] \). Suppose that in the \( i \)th parliament the party \( J(i) \) is chosen to form a government coalition, and it forms a winning coalition in the game \( G([\Lambda'(i)], V(i), q^{*}(\Lambda'(i), V(i))) \), where \( J(i) \) is included. Let \( \pi^C(i) \) be the probability that the coalition formed will be dominated by \( J(i) \) in the game \( G([\Lambda'(i)], V(i), q^{*}(\Lambda'(i), V(i))) \).

I now formulate the hypotheses:

\((H0^{COAL}) \) For every \( i \in [M'] \), \( \pi^C(i) = \pi^C(i) \).

\((H1^{COAL}) \) There exists \( i \in [M'] \) such that \( \pi^C(i) > \pi^C(i) \).

I use a statistical test similar to the one used in Section 3.

Consider \( M' \) Bernoulli trials with parameters \( \pi^C(1), \ldots, \pi^C(M') \). Let \( Z(\pi^C) \) be a random variable of the number of successes in these trials. The hypothesis \((H0^{COAL}) \) will be rejected and the hypothesis \((H1^{COAL}) \) will be accepted if \( \text{Prob}(Z(\pi^C) \geq D(C)) < 0.01 \).

Denote

\[
\mu^C(V) = \sum_{i=1}^{M'} \pi^C(i),
\]

\[
\sigma^C(V) = \sqrt{\sum_{i=1}^{M'} \pi^C(i)(1 - \pi^C(i))}.
\]

I use

\[
\frac{Z(\pi^C) - \mu^C(V)}{\sigma^C(V)} \xrightarrow{D} \mathcal{N}(1, 0), \text{ as } M' \to \infty \quad (4.1)
\]

to estimate \( \text{Prob}(Z(\pi^C) \geq D(C)) \).
For the data used (see Appendix 1 for a description), \( M' = 129, D(C) = 90, \mu^C(V) = 47.9478, \sigma^C(V) = 4.9184 \). From (4.1),

\[
Prob(Z(\pi^C) \geq D(C)) \approx 0.
\]

Therefore, hypothesis \((H_0^{COAL})\) must be rejected and \((H_1^{COAL})\) is accepted.

5 Number of parties and the existence of dominant player

I next examine the statistical relation of the existence a dominant party to the number of parties.

I consider the empirical data about the composition of parliaments (see Appendix 1). The frequency of a dominant party, relative to the number of parties in a parliament is shown in Figure 1

![Figure 1: Frequency of existence of a dominant party relative to the number of parties in a parliament](image)

I estimate the quadratic probit regression model with standard robustness. The dependent variable is the existence of a dominant player, while the independent variables
are the number of parties and the country from which an observation comes (dummy variable).\textsuperscript{1} Recall that there are 336 observations in the dataset.

Results of the regression model estimation are given in Table 1.

<table>
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<th>variable</th>
<th>coeff.</th>
<th>p-value</th>
<th>coeff.</th>
<th>p-value</th>
</tr>
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<td>$-0.0066$</td>
<td>0.022</td>
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<tr>
<td>Country Fixed Effect</td>
<td>No</td>
<td></td>
<td>Yes</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Quadratic probit regression. Dependent variable: existence of dominant party

There is only one observation where the number of parties is 17 (the maximal number of parties in the dataset). It can be referred to as an extreme observation. If it is excluded from the analysis, the results of the estimation of the quadratic probit regression model become less significant (see Table 2).

<table>
<thead>
<tr>
<th>variable</th>
<th>coeff.</th>
<th>p-value</th>
<th>coeff.</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(num. of parties)$^2$</td>
<td>$-0.023$</td>
<td>0.002</td>
<td>$-0.005$</td>
<td>0.082</td>
</tr>
<tr>
<td>num. of parties</td>
<td>$0.45$</td>
<td>0</td>
<td>$0.118$</td>
<td>0.032</td>
</tr>
<tr>
<td>Country Fixed Effect</td>
<td>No</td>
<td></td>
<td>Yes</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Quadratic probit regression, without the extreme observation. Dependent variable: existence of a dominant party

Next, I estimate the logarithmic probit regression model without including the extreme observation, when the country fixed effect is taken into account. The results, given in Table 3 are more significant than those of the quadratic regression model.

<table>
<thead>
<tr>
<th>variable</th>
<th>coeff.</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Log(num. of parties)</td>
<td>$0.286$</td>
<td>0.012</td>
</tr>
</tbody>
</table>

Table 3: Logarithmic probit regression, without the extreme observation. Dependent variable: existence of a dominant party

\textsuperscript{1}I will show results with and without country fixed effect.
All estimations above support the conjecture that when the number of parties is relatively low, the probability of a dominant party increases with the number of parties. The probability of the presence of a dominant party when the number of parties is high is less clear. I still conjecture that this probability decreases if the number of parties is sufficiently high, but there are not enough real-data observations to confirm it.

6 Acknowledgments

I would like to thank Bezalel Peleg for his contribution to definition of concepts of this paper. I would also like to thank Pavel Jehnov for a valuable discussion and for his assistance in the statistical analysis.

References


Appendix 1. Data description

I use the composition of parliaments (lower chambers) in 15 countries. I consider countries where and election years when the electoral system allocates seats to parties in (approximate) proportion to their public support.\footnote{In Belgium, after 1965 major parties spitted on the linguistic basis. To avoid confusion, I will not include data from this period in the analysis} In Table 4 countries and election years are summarized.

<table>
<thead>
<tr>
<th>Country</th>
<th>Years</th>
</tr>
</thead>
<tbody>
<tr>
<td>Austria</td>
<td>1920-1930,1945-2009</td>
</tr>
<tr>
<td>Belgium</td>
<td>1900-1939,1946-1965</td>
</tr>
<tr>
<td>Denmark</td>
<td>1920-2011</td>
</tr>
<tr>
<td>Federal Republic of Germany</td>
<td>1949-2009</td>
</tr>
<tr>
<td>Finland</td>
<td>1907-1939,1945-2007</td>
</tr>
<tr>
<td>Iceland</td>
<td>1959-2009</td>
</tr>
<tr>
<td>Israel</td>
<td>1949-2009</td>
</tr>
<tr>
<td>Italy</td>
<td>1946-1992</td>
</tr>
<tr>
<td>Luxembourg</td>
<td>1919-1937,1945-2009</td>
</tr>
<tr>
<td>Netherlands</td>
<td>1918-1937,1946-2010</td>
</tr>
<tr>
<td>Norway</td>
<td>1921-1936,1945-2009</td>
</tr>
<tr>
<td>Portugal</td>
<td>1975-2009</td>
</tr>
<tr>
<td>Sweden</td>
<td>1908-2010</td>
</tr>
<tr>
<td>Switzerland</td>
<td>1919-2007</td>
</tr>
<tr>
<td>Weimar Republic</td>
<td>1919-1933</td>
</tr>
</tbody>
</table>

Table 4: Countries and years of election

The sources of the data are Mackie and Rose [1991] and Müller and Strom [2003]. For data of recent years that is absent in those references, I use data from The European Journal of Political Research Political Data Yearbook (see references).
For the study of government coalitions, I use the data from the countries in Table 4, when it was available (there is no data for government coalitions in Iceland and Switzerland, and only partial data for the pre-World War II years in other countries). I also considered coalitions in Portugal only after 1980, since before then governments were appointed by the President, and not necessarily with parliamentary support.

I consider only majority coalitions in parliaments where a dominant party exists but it is not a dictator and it forms the government coalition.


The count of changes of cabinets is in accordance with the count used in the sources, except for changes in the cabinets of Israel that were counted manually. General condi-
tions for a definition of a new cabinet, according to Müller and Strom [2003], were:

1. The coalition was formed after an election
2. The composition of the coalition party was changed
3. A new prime minister was appointed.

Appendix 2: Parameters for Bernoulli trials and summary of findings

We exclude from the analysis $n = 2$. Obviously, for $n = 2$ the dictator always exists, therefore statistical analysis is meaningless.

The calculation of $\pi^{\text{DOM}}(3)$ and $\pi^{\text{DIC}}(3)$ is straightforward. Note that for $n = 3$ a party is dominant iff it is the dictator.

For $n > 3$ $\pi^{\text{DOM}}(n)$ and $\pi^{\text{DIC}}(n)$ are estimated by the Monte-Carlo simulation. Recall,

$$\Omega(n) = \{(w_1, \ldots, w_n)|\sum_{i=1}^n w_i\}$$

is $(n - 1)$-dimensional simplex,

$$\Omega^{\text{DOM}}(n) = \{w \in \Omega| \text{ in the game } G([n], w, q^*(n, w)) \text{ there is a dominant player}\},$$

and

$$\Omega^{\text{DIC}}(n) = \{w \in \Omega| \text{ in the game } G([n], w, q^*(n, w)) \text{ there is a dictator}\}.$$ 

The simulation chooses random vectors $(w_1, \ldots, w_n)$ from $\Omega(n)$ with uniform distribution, and counts vectors $(w_1, \ldots, w_n) \in \Omega^{\text{DOM}}(n)$ and $(w_1, \ldots, w_n) \in \Omega^{\text{DIC}}(n)$. The simulation makes 1000000 trials for $n < 9$, 10000 trials for $10 \leq n \leq 13$ and 1000 for $14 \leq n \leq 17$.

To produce a random vector $(w_1, \ldots, w_n)$ from $\Omega(n)$ with uniform distribution, we use a well-known fact (see Feller [1971]) that if $W_1^*, \ldots, W_n^*$ are i.i.d. with exponential distribution, then $(\frac{W_1^*}{\sum_1^n W_i^*}, \ldots, \frac{W_n^*}{\sum_1^n W_i^*})$ is uniformly distributed on $\Omega(n)$. 

21
<table>
<thead>
<tr>
<th>Number of parties $(n)$</th>
<th>$\pi^{DOM}(n)$</th>
<th>$\pi^{DIC}(n)$</th>
<th>Number of observations</th>
<th>Number of dominants players</th>
<th>Number of dictators</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.75</td>
<td>0.75</td>
<td>21</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>0.75</td>
<td>0.5</td>
<td>36</td>
<td>27</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>0.78</td>
<td>0.31</td>
<td>64</td>
<td>51</td>
<td>8</td>
</tr>
<tr>
<td>6</td>
<td>0.8</td>
<td>0.19</td>
<td>53</td>
<td>40</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>0.78</td>
<td>0.1</td>
<td>39</td>
<td>32</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>0.75</td>
<td>0.06</td>
<td>27</td>
<td>22</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>0.7</td>
<td>0.04</td>
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<td>24</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>0.62</td>
<td>0.02</td>
<td>20</td>
<td>16</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>0.54</td>
<td>0.01</td>
<td>13</td>
<td>12</td>
<td>0</td>
</tr>
<tr>
<td>12</td>
<td>0.46</td>
<td>0.006</td>
<td>13</td>
<td>9</td>
<td>0</td>
</tr>
<tr>
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<td>0.003</td>
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<tr>
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<tr>
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<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 5: Parameters for Bernoulli trials, and summary of findings